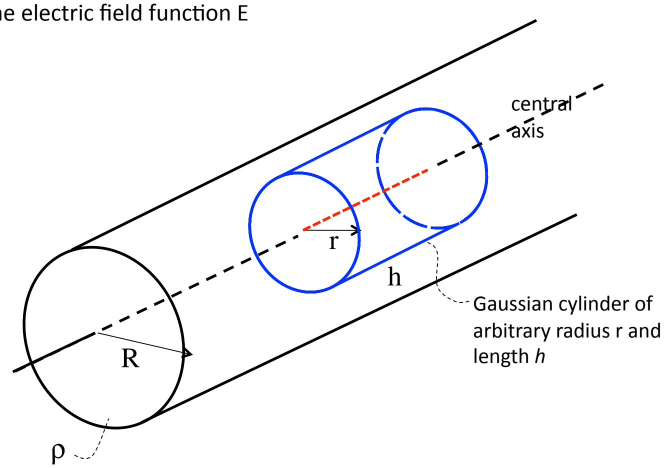
Problem 24.33

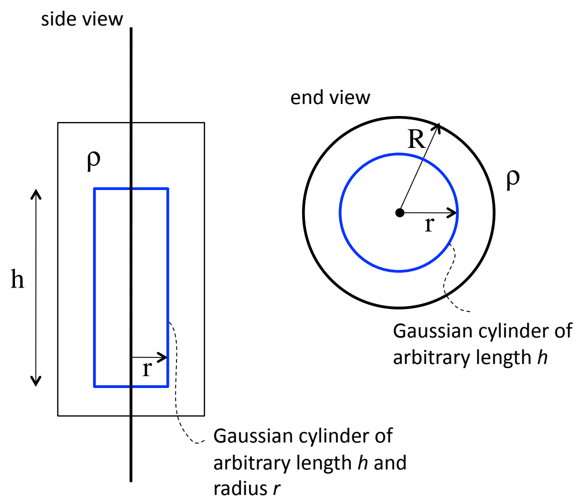
A cylinder of radius R has a constant charge distribution ρ shot through it. Derive an expression for the electric field function E (r) when r <R.

For the 3-d view:



CROSS-SECTIONS

A cross-sectional sketch of the cylinder is shown to the right.



THE EASY WAY: Because the charge density is constant, the amount of charge inside the Gaussian surface of arbitrary length "h" will be:

$$q_{encl} = \rho V_{encl} = \rho\left[\pi r^2 (h)\right]$$

Noting that all of the flux will pass through the cylindrical part of the Gaussian surface with none through the end-caps, Gauss's Law yields:

$$\int_A \vec{E} \bullet d\vec{A} = \frac{q_{enclose}}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi r h) = \frac{\rho \pi r^2 h}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\rho}{2\varepsilon_o} r$$

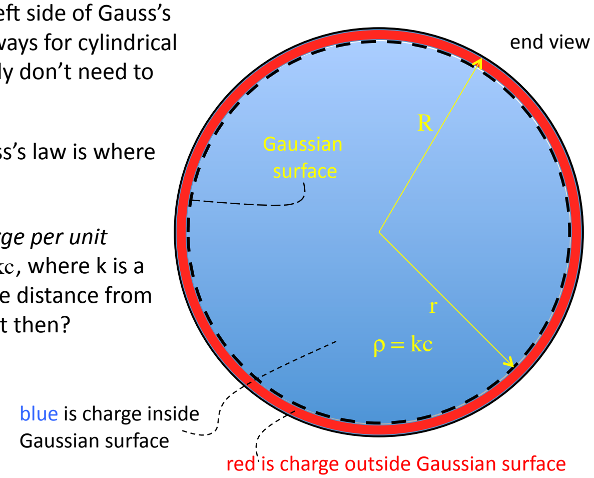
This would clearly be the way to do this problem if it was found on an AP test and time was a problem. If, on the other hand (and additionally), you had time and the volume charge density wasn't constant, how would you do the problem then?

THE HARD PROBLEM: What happens if the charge isn't uniformly distributed? In that case, determining the "charge enclosed" inside the Gaussian surface gets tricky. If you have intellectual curiosity as to how to do a problem like this, read on. Otherwise, stop here.

In this problem, the left side of Gauss's law is the same as always for cylindrical symmetry, so we really don't need to mess with that.

The right side of Gauss's law is where being clever pays off.

Let's assume the *charge per unit volume* varies as $\rho = kc$, where k is a constant and *c* is some distance from the central axis. What then?



To deal with this, consider a differentially thin cylindrical shell of radius "c" (where c < r), of differential thickness "dc," of differential volume "dV" and of arbitrary length h (i.e., the length of the Gaussian cylinder).

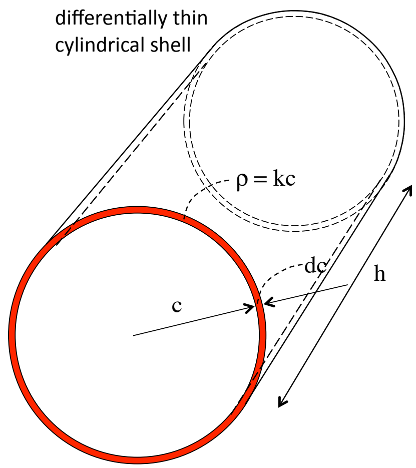
If we knew dV for the shell, we could multiply it by ρ to get the charge enclosed *within the shell*.

dV equals the circumference of the circle ($2\pi c$) times its thickness (dc) times its length (h). That is:

$$dV = (2\pi hc)\,dc$$

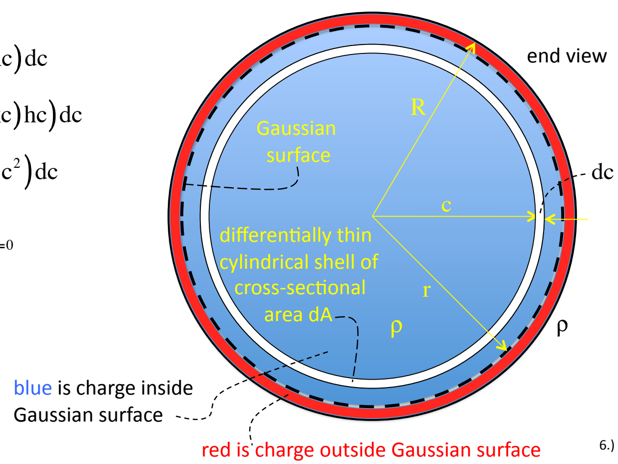
That means the differential charge dq inside the shell is:

$$dq = \rho \quad dV$$
$$= (kc)\left[(2\pi hc)\,dc\right]$$



The sketch below gives an end-view of the situation, complete with shell. Clearly, the total charge inside the Gaussian surface will be the sum of all of the dq's from zero to the Gaussian radius *r*. That is:

$$q_{encl} = \int dq$$
$$= \int_{c=0}^{r} (2\pi\rho hc)\,dc$$
$$= \int_{c=0}^{r} (2\pi(kc)hc)\,dc$$
$$= 2\pi kh \int_{c=0}^{r} (c^2)\,dc$$
$$= 2\pi kh \left.\frac{c^3}{3}\right|_{c=0}^{r}$$
$$= \frac{2\pi kh}{3} r^3$$



Using this information in Gauss's Law yields:

$$\int_A \vec{E} \bullet d\vec{A} = \frac{q_{enclose}}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi rh) = \frac{\int_{c=0}^{r} (2\pi(kc)hc)\,dc}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi rh) = \frac{2\pi kh \int_{c=0}^{r} (c^2)\,dc}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi rh) = \frac{2\pi kh \left.\frac{c^3}{3}\right|_{c=0}^{r}}{\varepsilon_o}$$

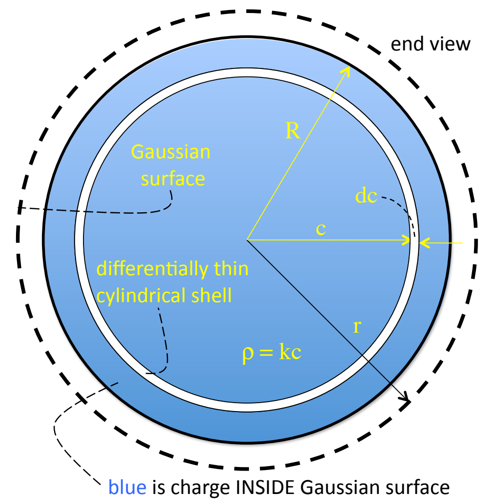
$$\Rightarrow \quad E(2\pi rh) = \frac{\frac{2\pi kh}{3} r^{3}}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{k}{3\varepsilon_o} r^2$$

This works for r < R. How about for r > R?

When the Gaussian radius (i.e., the radius at which you are evaluating the electric field function) and the radius in which the charge is enclosed (*r* in the previous case) is the same, your Gauss's Law expression will have *r* terms that will cancel.

When the Gaussian radius *r* is outside the structure, the "charge enclosed" doesn't go all the way out to the Gaussian surface, so the integral's limits change from *r* to *R* (the edge is where the charge stops. That calculation is shown on the next page.



For this case, Gauss's Law yields:

$$\int_A \vec{E} \bullet d\vec{A} = \frac{q_{enclose}}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi rh) = \frac{\int_{c=0}^{R} (2\pi(kc)hc)dc}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi rh) = \frac{\frac{2\pi kh}{3}R^3}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{k}{3\varepsilon_o}\frac{R^3}{r}$$

Kindly note that as you would expect *E* to go to zero as *r* goes to infinity, that is exactly what the expression suggests. It all works nicely.